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Abstract: We show that the problem of finding a set with maximum cohesion in an undirected network is **NP**-hard.

Key-words: social networks, complex networks, cohesion, np-complete, complexity

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Maximiser la Cohésion est NP-dur

Résumé : Nous montrons que le problème de trouver un ensemble de cohésion maximum dans un graphe non orienté est **NP**-dur.

Mots-clés : réseaux sociaux, réseaux complexes, coésion, np-complet, compléxité

Introduction

In [1], we have introduced a new metric called the *cohesion* which rates the community-ness of a group of people in a social network from a sociological point of view. Through a large scale experiment on Facebook, we have established that the cohesion is highly correlated to the subjective user perception of the communities. In this article, we show that finding a set of vertices with maximum cohesion is **NP**-hard.

Notations

Let $G = (V, E)$ be a graph with vertex set V and edge set E of size $n = |V| \geq 4$. For all vertices $u \in V$, we write $d_G(u)$ the degree of u, or more simply $d(u)$[1]. A *triangle* in G is a triplet of pairwise connected vertices.

For all sets of vertices $S \subseteq V$, let $G[S] = (S, E_S)$ be the subgraph induced by S on G. We write $m(S) = |E_S|$ the number of edges in $G[S]$, and $i(S) = |\{(u, v, w) \in S^3 : (uv, vw, uw) \in E^3\}|$ the number of triangles in $G[S]$. We define $o(S) = |\{(u, v, w), (u, v) \in S^2, w \in V \setminus S : (uv, vw, uw) \in E^3\}|$, the number of *outbound* triangles of S, that is: triangles in G which have exactly two vertices in S.

Moreover, for all (u, v) in E, let $\triangle(uv) = |\{w \in V : (uw, vw) \in E^2\}|$ be the number of triangles the edge uv belongs to in G.

Finally, we recall the definition of the cohesion of a set S in G:

$$\mathcal{C}(S) = \frac{i(S)^2}{\binom{|S|}{3}(i(S) + o(S))}$$

An example is given on Figure 1. The cohesion of a given set S in G can naively be computed in $\mathcal{O}(n^3)$ by listing all triangles in G and counting those inside and outbound to S.

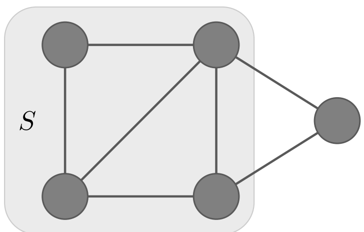

Figure 1: In this example, $i(S) = 2$ and $o(S) = 1$, thus $\mathcal{C}(S) = \frac{1}{6}$

In this article we examine the problem of finding a set of vertices $S \subseteq V$ of maximum cohesion, i.e. for all subset $S' \subseteq V$, $\mathcal{C}(S') \leq \mathcal{C}(S)$.

Outline

We now proceed to prove that finding a set of vertices with maximum cohesion in G is **NP**-hard. We will first show in Section 1 that this problem is equivalent

[1] Here, as elsewhere, we drop the index referring to the underlying graph if the reference is clear.

to that of finding a connected set of vertices with maximum cohesion in G. The decision problem associated to the latter is CONNECTED-COHESIVE.

Then, we shall prove that CONNECTED-COHESIVE is **NP**-complete by reducing CLIQUE (problem GT19 in [2]). From there we deduce that the optimization problem of finding a set of vertices with maximum cohesion is **NP**-hard.

Problems

1. CONNECTED-COHESIVE:

Input	A graph $G = (V, E)$, $\lambda \in \mathbb{Q}$, $\lambda \in [0, 1]$
Question	Is there a subset connected S of V such that $\mathcal{C}(S) \geq \lambda$?

2. CLIQUE:

Input	A graph $G = (V, E)$, $k \in \mathbb{N}, k \leq \lvert V \rvert$
Question	Is there a subset S of V such that $\lvert S \rvert = k$ and the subgraph induced by S is a clique?

1 A maximum cohesive group is connected

In order to prove that a set of vertices with maximum cohesion in a given network is connected, we need the following lemma:

Lemma 1.1. *Let $S_1 \subseteq V$ and $S_2 \subseteq V$ be two disconnected sets of vertices ($(S_1 \times S_2) \cap E = \emptyset$). If $\mathcal{C}(S_1) \leq \mathcal{C}(S_1 \cup S_2)$ then $\mathcal{C}(S_2) > \mathcal{C}(S_1 \cup S_2)$.*

Proof. Suppose $\mathcal{C}(S_1) \leq \mathcal{C}(S_1 \cup S_2)$ and $\mathcal{C}(S_2) \leq \mathcal{C}(S_1 \cup S_2)$. Given that S_1 and S_2 are disconnected, $i(S_1 \cup S_2) = i(S_1) + i(S_2)$ and $o(S_1 \cup S_2) = o(S_1) + o(S_2)$. We can then write:

$$\frac{i(S_1)^2}{\binom{|S_1|}{3}} \leq (i(S_1) + o(S_1))\mathcal{C}(S_1 \cup S_2) \tag{1}$$

$$\frac{i(S_2)^2}{\binom{|S_2|}{3}} \leq (i(S_2) + o(S_2))\mathcal{C}(S_1 \cup S_2) \tag{2}$$

By summing (1) and (2), we obtain:

$$\begin{aligned}
\frac{i(S_1)^2}{\binom{|S_1|}{3}} + \frac{i(S_2)^2}{\binom{|S_2|}{3}} &\leq (i(S_1) + o(S_1) + i(S_2) + o(S_2))\mathcal{C}(S_1 \cup S_2) \\
&\leq (i(S_1 \cup S_2) + o(S_1 \cup S_2))\mathcal{C}(S_1 \cup S_2) \\
&\leq \frac{(i(S_1) + i(S_2))^2}{\binom{|S_1|+|S_2|}{3}}
\end{aligned}$$

Furthermore, given that $|S_1|, |S_2| > 1$,

$$\binom{|S_1|}{3} + \binom{|S_2|}{3} < \binom{|S_1| + |S_2|}{3}$$

We then have:

$$\frac{i(S_1)^2}{\binom{|S_1|}{3}} + \frac{i(S_2)^2}{\binom{|S_2|}{3}} < \frac{(i(S_1)+i(S_2))^2}{\binom{|S_1|}{3}+\binom{|S_2|}{3}}$$

Which simplifies to:

$$\left(\binom{|S_2|}{3} i(S_1) - \binom{|S_1|}{3} i(S_2)\right)^2 < 0$$

Hence the contradiction. Therefore, for all $S_1, S_2 \subseteq V$, disconnected:

$$\mathcal{C}(S_1) \leq \mathcal{C}(S_1 \cup S_2) \Rightarrow \mathcal{C}(S_2) > \mathcal{C}(S_1 \cup S_2)$$ □

Theorem 1.2. *Let S be the set of vertices of G with the highest cohesion, S is connected.*

Proof. Suppose S is not connected, then their exist two disconnect subsets $S_1, S_2 \subseteq S$ such that $S = S_1 \cup S_2$. Given that S has maximum cohesion, we have $\mathcal{C}(S) \geq \mathcal{C}(S_1)$. Thus per Lemma 1.1: $\mathcal{C}(S) < \mathcal{C}(S_2)$ and S does not have the highest cohesion, hence the contradiction. □

Corollary 1.3. *Per Theorem 1.2, the problem of searching for a set of vertices with maximum cohesion is strictly equivalent to that of searching a set of connected vertices with maximum cohesion.*

2 Connected-Cohesive is NP-complete

First note that given a set S of vertices of G, it is possible to verify that S is a solution of Connected-Cohesive by computing its cohesion, its size, its connectivity and the minimum degree of its vertices, all in polynomial time. Therefore Connected-Cohesive is in **NP**.

Algorithm 1 Transforms an instance of Clique in an instance of Connected-Cohesive

Require: $G = (V, E), k \in \mathbb{N}$

```
W := ∅
E' := E
for uv ∈ V² \ E do
    let K be a clique of size 2 binom(n,3)^4
    W ← W ∪ K
    E' ← E' ∪ {uv} ∪ ({u, v} × K)
end for
return G' = (V ∪ W, E'), λ = binom(k,3) / (binom(k,3) + binom(k,2)(n−k))
```

(Line 4: size $2\binom{n}{3}^4$; line 8: $\lambda = \frac{\binom{k}{3}}{\binom{k}{3}+\binom{k}{2}(n-k)}$.)

Let us now reduce Clique to Connected-Cohesive. Let $(G = (V, E), k \in \mathbb{N})$ be an instance of Clique[2]. We can assume that G is connected (if not, we

[2]We consider here that $|G| > 2$ and $k > 2$, although this is not exactly Clique, this problem is clearly **NP**-complete, given that the complexity of Clique does not arise from those small values.

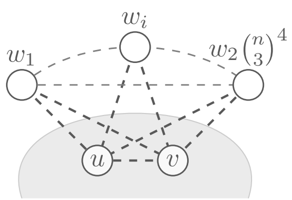

Figure 2: Illustration of Algorithm 1. At this step, wejoin u and v, add a clique of size $2\binom{n}{3}^4$ to the network, and join u and v to all vertices in the added clique.

use the following reasoning separately on each connected component of G). We construct an instance $(G' = (V', E'), \lambda)$ of Connected-Cohesive by adding an edge between all non connected vertices u and v in G and then linking those two vertices to all vertices in a clique of size $2\binom{n}{3}^4$which we add to the network, as described in Algorithm 1 and illustrated by Figure 2.

Theorem 2.1. *There exist a clique of size k in G* iff *there exist a connected group of vertices of G' with cohesion $\lambda \geq \frac{\binom{k}{3}}{\binom{k}{3}+\binom{k}{2}(n-k)}$.*

Proof. Let $K \subseteq V$, be a clique of size $|K| = k$ in G. Given that no node or edges are deleted when constructing G', G is a subgraph of G' and thus K is a clique in G' and $i_{G'}(K) = \binom{k}{3}$.

Moreover, by construction, $G'[V]$ is a clique and for all u un K, the neighbors of u are also in V. Therefore, each edge in K forms one triangle with each vertex in $V \setminus K$, which leads to $o_{G'}(K) = \binom{k}{2}(n-k)$. Finally, this gives a cohesion:

$$\mathcal{C}_{G'}(K) = \frac{\binom{k}{3}}{\binom{k}{3} + \binom{k}{2}(n-k)}$$

Conversely, let $S \subseteq V'$ be a connected set of vertices such that $\mathcal{C}_{G'}(S) \geq \frac{\binom{k}{3}}{\binom{k}{3}+\binom{k}{2}(n-k)}$. We will show that S is a clique of size larger than k and that $S \subseteq V$. First note that $|S| \geq 3$, because by definition, if $|S| < 3$, $\mathcal{C}_{G'}(S) = 0$ which would lead to a contradiction.

First, suppose that S is not a clique in G, then let us distinguish two cases:

1. If $S \subseteq V$ and S is not a clique, then S contains two vertices $u, v \in V^2$ such that $uv \notin E$.

2. If $S \not\subseteq V$, then $\exists u \in S \setminus V$, and S being connected, there exist $v \in V'$ such that $uv \notin E$.

Therefore, if S is not a clique in G, it contains an edge $uv \notin E$ and by construction, this edge belongs to at least $2\binom{n}{3}^4$ triangles, which leads to:

$$\begin{aligned} i_{G'}(S) + o_{G'}(S) \geq K \\ \mathcal{C}_{G'}(S) &\leq \frac{i_{G'}(S)^2}{2\binom{|S|}{3}\binom{n}{3}^4} \\ &\leq \frac{1}{2\binom{n}{3}^2} \\ &< \frac{\binom{k}{3}}{\binom{k}{3} + \binom{k}{2}(n-k)} \end{aligned}$$

Hence the contradiction, therefore S must be a clique in G. From there it comes that:

$$\mathcal{C}_{G'}(S) = \frac{\binom{k'}{3}}{\binom{k'}{3} + \binom{k'}{2}(n-k')}$$

where $k' = |S|$. Therefore:

$$\begin{aligned} \mathcal{C}_{G'}(S) \geq \frac{\binom{k}{3}}{\binom{k}{3} + \binom{k}{2}(n-k)} &\Leftrightarrow \frac{\binom{k'}{2}(n-k')}{\binom{k'}{3}} \leq \frac{\binom{k}{2}(n-k)}{\binom{k}{3}} \\ &\Leftrightarrow \frac{n-k'}{k'-3} \leq \frac{n-k}{k-3} \\ &\Leftrightarrow k' \geq k \end{aligned}$$

Therefore, we can now conclude that if there exist a connected set S in G' with cohesion $\mathcal{C}_{G'}(S) \geq \frac{\binom{k}{3}}{\binom{k}{3}+\binom{k}{2}(n-k)}$, then S is a clique of size at least k in G, and thus there exist a clique $K \subseteq S$ of size k in G. □

Theorem 2.2. Connected-Cohesive *is **NP**-complete.*

Proof. Per Theorem 2.1, there exist a clique of size k in G *iff* there exist a connected subset of vertices of G' of cohesion $\lambda \geq \frac{\binom{k}{3}}{\binom{k}{3}+\binom{k}{2}(n-k)}$ and the transformation from G, k to G', λ runs in polynomial time. Thus Clique is reducible to Connected-Cohesive and Connected-Cohesive is **NP**-hard.

Given that Connected-Cohesive is in **NP**, the problem is thus **NP**-complete. □

3 Conclusion

The associated decision problem being **NP**-complete, the problem of finding a set of vertices with maximum cohesion is **NP**-hard[3].

[3]Note that the problem of finding a set of vertices of maximum cohesion containing a set of predefined vertices is also **NP**-hard, by an immediate reduction

References

[1] Adrien Friggeri, Guillaume Chelius, and Eric Fleury. Triangles to Capture Social Cohesion. In *Third IEEE International Conference on Social Computing*, Cambridge, United States, September 2011.

[2] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman, San Francisco, 1979.

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